International Journal of Advanced and Applied Sciences, 5(12) 2018, Pages: 16-24

Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

Application of the reduced mechanism method to determine the law of motion of a planar mechanism with multiple degrees of freedom

Avdo Voloder, Elvedin Kljuno *

Mechanical Engineering Faculty, University of Sarajevo, Sarajevo, Bosnia and Herzegovina

ARTICLE INFO

Article history:
Received 28 May 2018
Received in revised form
1 September 2018
Accepted 16 September 2018

Keywords:
Planar mechanism
Angular acceleration
Mechanism reduction determinacy
Partial mechanism reduction

ABSTRACT

This paper shows a method to determine unknown angular accelerations of driving members of a planar mechanism with multiple degrees of freedom via partial mechanism reduction, assuming that driving loads are known for those driving members. Besides the partial reduction of mechanism, here we use the analysis of primary and secondary accelerations, as well as the principle of virtual displacements (virtual work). Using this method, a set of decoupled equations is obtained, which is an advantage when compared to classical methods, such as an application of generalized laws of dynamics, which result in a set of equations that are coupled. As an illustration of how to use the described method, an example is shown.

1. Introduction

As far as the problem of multibody system dynamics is concerned, there are several usual methods (Gernet, 1973; Hibbeler, 2013; Johnston et al., 2009). The method of the reduced mechanism is suitable for problems of planar mechanisms with single degree of freedom (DOF), as shown in Ilic (1968). However, the idea is to find a way to expand the single DOF method to be applicable for multiple DOF planar mechanisms.

1.1. Definition of a reduced mechanism

By drawing members of a single DOF planar bar mechanism parallel to its real positions and orientations (pose) using a scale (reduction) factor, while member centers of rotation (poles) are placed into a single point and connection points between members are maintained, a reduced mechanism of the (real) mechanism is obtained (Ilic, 1965). Fig. 1 shows a four-bar mechanism (four-joint mechanism) along with a corresponding reduced mechanism. In this case, the pole P^* of the reduced mechanism coincides with points A^* and E^* that correspond to poles A and E of members 1 and 3 of the real mechanism.

It is obvious that proportions between particular members of the real mechanism and their corresponding members of the reduced mechanism are generally not same.

The scaling factor of the member (j) is defined as a ratio between the vector of **j**-th member of the reduced mechanism and the vector of **j**-th member of the real mechanism, which are parallel, as in Fig. 2. (Ilic, 1968).

$$\mu_j = \frac{\vec{L}_j^*}{\vec{L}_j} = \frac{\overrightarrow{Q_j^* W_j^*}}{\overrightarrow{Q_j W_j}}, \qquad (1)$$

where,
Q_j, W_j – are the initial and final point of the member (j) of the real mechanism,
Q_j^*, W_j^* – are the initial and final point of the member (j) of the reduced mechanism.

Then, for an arbitrary point K_{ji} (i-th point of the j-th mechanism member)

$$\mu_j = \frac{\vec{p}_{ji}^*}{\vec{p}_{ji}} = \frac{\overrightarrow{P^* K_{ji}^*}}{\overrightarrow{P_j K_{ji}}}, \qquad (2)$$

where,
$\vec{p}_{ji}$ - vector between instantaneous pole P_j of member (j) and the point K_{ji} that member,
$\vec{p}_{ji}^*$ - vector between pole P^* of the reduced mechanism and the point K_{ji}^* of member (j).

Factor of reduction can have any sign and it can be zero, as well. The ratio of angular velocity of any member and its factor of reduction remains the same for all mechanism members (Hufnagl, 1984)

$$\frac{\vec{\omega}_j}{\mu_j} = \frac{\vec{\omega}_{j+1}}{\mu_{j+1}} = \frac{\vec{\omega}_1}{\mu_1}, \qquad (3)$$

* Corresponding Author.
Email Address: kljuno@mef.unsa.ba (E. Kljuno)
https://doi.org/10.21833/ijaas.2018.12.003

and the velocity of the point K_{ji} is

$$\vec{v}_{ji} = [\vec{\omega}_j, \vec{p}_{ji}] = \left[\frac{\vec{\omega}_j}{\mu_j}, \vec{p}^*_{ji}\right]. \tag{4}$$

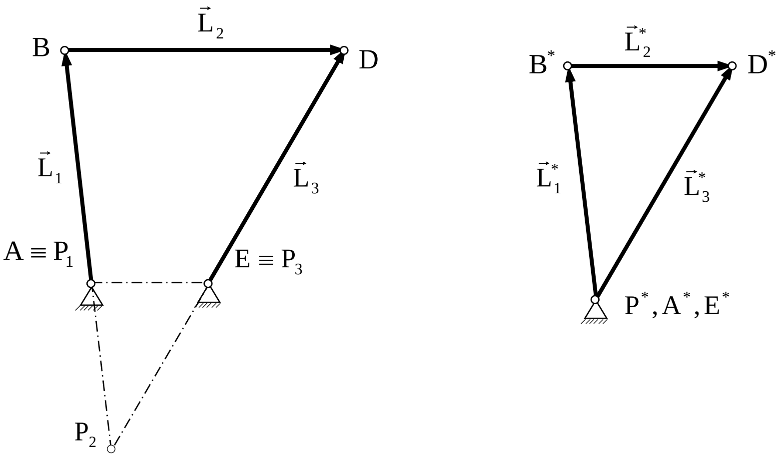

Fig. 1: Mechanism with a single DOF and its unique reduced mechanism

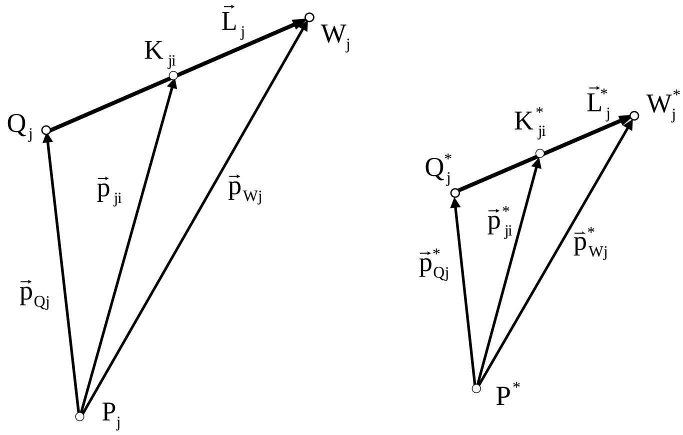

Fig. 2: The relationship between corresponding vectors of the real mechanism members and the reduced mechanism members

As far as the translational kinematic pair is concerned, the slider of that pair is replaced by a virtual (relative) member, which is perpendicular to the direction of the relative motion (sliding) over the guide (the slider B at the Fig. 3a is replaced by virtual member B_1B_3 at the Fig. 3b). Further, the reduced mechanism (Fig. 3c) is constructed based on the virtual mechanism at the Fig. 3b (Hufnagl, 1984).

As the slider has the angular velocity equal to the angular velocity of the slider guide, then the factors of their reduction are equal.

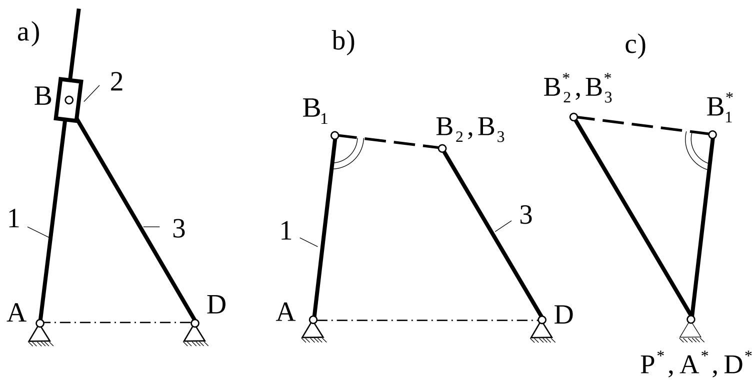

Fig. 3: Replacement of the translational kinematic pair (slider-guide) by a virtual member

2. Partial reduction of mechanism

The problem is how to define a unique reduced mechanism of a real mechanism with multiple DOF. Let us assume that we have such a mechanism in the Fig. 4.

If we construct a corresponding integral reduced mechanism in compliance with the aforementioned definition of the reduced mechanism, then the solution would not be unique, there would be infinitely many solutions (Fig. 5).

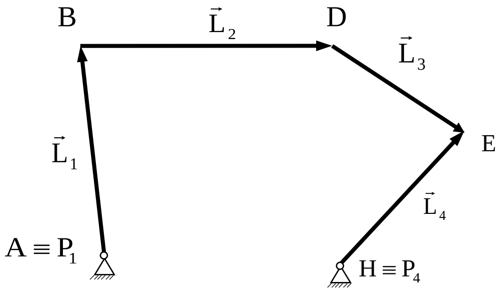

Fig. 4: Mechanism with two DOF

If we construct a partial reduced mechanism, by blocking all degrees of freedom except one, then solutions would be determined and unique. To each DOF of the real mechanism, there would exist one reduced mechanism (Fig. 6).

3. Primary and secondary characteristics of the mechanism accelerations with multiple DOF

Let us assume that a planar mechanism has ***s*** degrees of freedom, such that the mechanism in general case has ***s*** independent drives (Fig. 7). Then, the angular velocity of the mechanism's **j**-th member can be expressed via angular velocities of driving members of the mechanism:

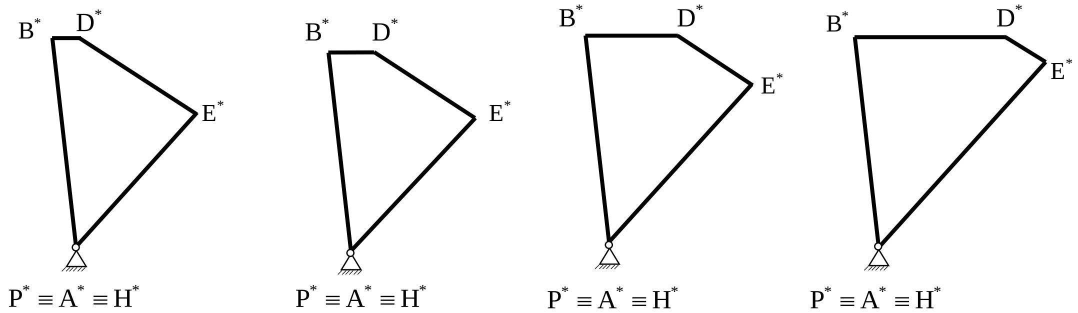

Fig. 5: Indeterminacy of the integral reduced mechanism of the real mechanism with multiple DOF

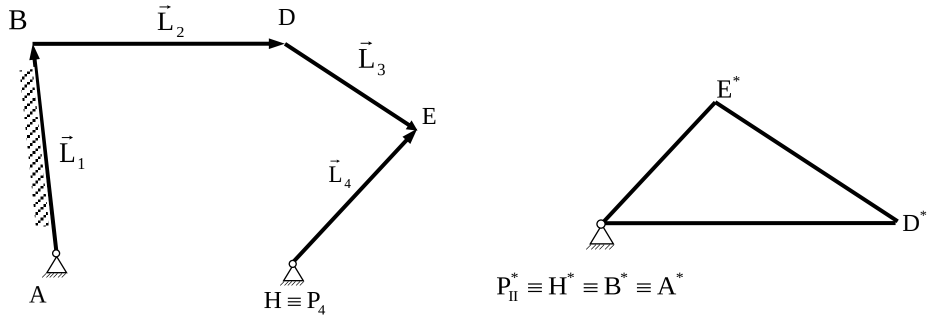

Fig. 6. Determinacy of partial reduced mechanisms that correspond to a real mechanism with multiple DOF

$$\vec{\omega}_j = \frac{d\phi_j}{dt}\vec{k} = \left(\frac{\partial\phi_j}{\partial\phi_1}\dot{\phi}_1 + \frac{\partial\phi_j}{\partial\phi_2}\dot{\phi}_2 + \cdots + \frac{\partial\phi_j}{\partial\phi_1}\dot{\phi}_1\right)\vec{k}, \qquad (5)$$

$$\vec{\omega}_j = \frac{d\phi_j}{dt}\vec{k} = \vec{k}\sum_q \frac{\partial\phi_j}{\partial\phi_q}\dot{\phi}_q = \sum_q \frac{\partial\phi_j}{\partial\phi_q}\vec{\omega}_q = \sum_q \vec{\omega}_{j,q}, \qquad (6)$$

where the **q**-th partial contribution of the angular velocity existing within the total angular velocity is

$$\vec{\omega}_{j,q} = \frac{\partial\phi_j}{\partial\phi_q}\dot{\phi}_q\vec{k}, \qquad (7)$$

where,

$\vec{k}$ - is the unit vector perpendicular to the plane of the mechanism,

ϕ_q, $\vec{\omega}_q$ - are the angle of rotation and the angular velocity of an independent driving member **q**.

Angular acceleration of the **j**-th member of the mechanism is

$$\vec{\epsilon}_j = \frac{d\vec{\omega}_j}{dt} = \left(\frac{\partial^2\phi_j}{\partial\phi_1^2}\dot{\phi}_1^2 + \frac{\partial^2\phi_j}{\partial\phi_2^2}\dot{\phi}_2^2 + \cdots + \frac{\partial^2\phi_j}{\partial\phi_s^2}\dot{\phi}_s^2\right)\vec{k} + \left(\frac{\partial\phi_j}{\partial\phi_1}\ddot{\phi}_1 + \frac{\partial\phi_j}{\partial\phi_2}\ddot{\phi}_2 + \cdots + \frac{\partial\phi_j}{\partial\phi_s}\ddot{\phi}_s\right)\vec{k}, \qquad (8)$$

i.e.

$$\vec{\epsilon}_j = \vec{k}\sum_q \frac{\partial^2\phi_j}{\partial\phi_q^2}\dot{\phi}_q^2 + \vec{k}\sum_q \frac{\partial\phi_j}{\partial\phi_q}\ddot{\phi}_q. \qquad (9)$$

The component of the angular acceleration, which depends on angular velocities of driving members, will be denoted as the primary angular acceleration of the j-th member

$$\vec{\epsilon}_j^{\omega} = \vec{k}\sum_q \frac{\partial^2\phi_j}{\partial\phi_q^2}\dot{\phi}_q^2 = \sum_q \vec{\epsilon}_{j,q}^{\omega}, \qquad (10)$$

where the **q**-th partial contribution of the primary angular acceleration to the total primary angular acceleration of the j-th member

$$\vec{\epsilon}_{j,q}^{\omega} = \frac{\partial^2\phi_j}{\partial\phi_q^2}\dot{\phi}_q^2\,\vec{k}, \qquad (11)$$

and the component that depends on angular accelerations of driving members will be named as the secondary angular acceleration of the j-th member

$$\vec{\epsilon}_j^{\epsilon} = \vec{k}\sum_q \frac{\partial\phi_j}{\partial\phi_q}\ddot{\phi}_q = \sum_q \vec{\epsilon}_{j,q}^{\epsilon}, \tag{12}$$

where the **q**-th partial contribution of the secondary angular acceleration to the total secondary angular acceleration is

$$\vec{\epsilon}_{j,q}^{\epsilon} = \frac{\partial\phi_j}{\partial\phi_q}\ddot{\phi}_q\,\vec{k}. \tag{13}$$

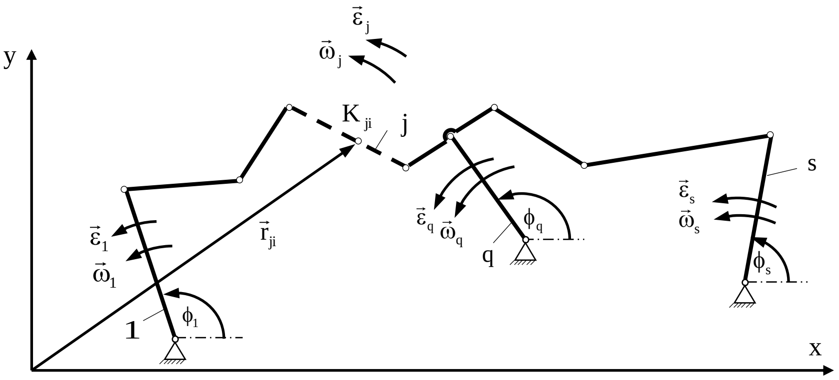

Fig. 7. The analysis of the primary and secondary acceleration characteristics of the mechanism with multiple DOF

Velocity of the point K_{ji} is

$$\vec{v}_{ji} = \frac{d\vec{r}_{ji}}{dt} = \left(\frac{\partial\vec{r}_{ji}}{\partial\phi_1}\dot{\phi}_1 + \frac{\partial\vec{r}_{ji}}{\partial\phi_2}\dot{\phi}_2 + \cdots + \frac{\partial\vec{r}_{ji}}{\partial\phi_s}\dot{\phi}_s\right)\vec{k} = \sum_q \frac{\partial\vec{r}_{ji}}{\partial\phi_q}\dot{\phi}_q = \sum_q \vec{v}_{ji,q}, \tag{14}$$

where,

$$\vec{v}_{ji,q} = \frac{\partial\vec{r}_{ji}}{\partial\phi_q}\dot{\phi}_q. \tag{15}$$

Acceleration of the point K_{ji} is

$$\vec{a}_{ji} = \frac{d\vec{v}_{ji}}{dt} = \frac{\partial^2\vec{r}_{ji}}{\partial\phi_1^2}\dot{\phi}_1^2 + \frac{\partial^2\vec{r}_{ji}}{\partial\phi_2^2}\dot{\phi}_2^2 + \cdots + \frac{\partial^2\vec{r}_{ji}}{\partial\phi_s^2}\dot{\phi}_s^2 + \frac{\partial\vec{r}_{ji}}{\partial\phi_1}\ddot{\phi}_1 + \frac{\partial\vec{r}_{ji}}{\partial\phi_2}\ddot{\phi}_2 + \cdots + \frac{\partial\vec{r}_{ji}}{\partial\phi_s}\ddot{\phi}_s\,, \tag{16}$$

i.e.

$$\vec{a}_{ji} = \sum_{q=1}^{s}\frac{\partial^2\vec{r}_{ji}}{\partial\phi_q^2}\dot{\phi}_q^2 + \sum_{q=1}^{s}\frac{\partial\vec{r}_{ji}}{\partial\phi_q}\ddot{\phi}_q. \tag{17}$$

Therefore, the acceleration of an arbitrary mechanism's point K_{ji} depends, in general, on angular velocities and angular accelerations of the driving members of the mechanism.

The first component of the acceleration represents the primary acceleration of the point K_{ji}

$$\vec{a}_{ji}^{\omega} = \sum_{q=1}^{s}\frac{\partial^2\vec{r}_{ji}}{\partial\phi_q^2}\dot{\phi}_q^2 = \sum_{q=1}^{s}\vec{a}_{ji,q}^{\omega}, \tag{18}$$

and the second one represents the secondary acceleration

$$\vec{a}_{ji}^{\epsilon} = \sum_{q=1}^{s}\frac{\partial\vec{r}_{ji}}{\partial\phi_q}\ddot{\phi}_q = \sum_{q=1}^{s}\vec{a}_{ji,q}^{\epsilon}. \tag{19}$$

Additionally,

$$\vec{a}_{ji,q}^{\omega} = \sum_{q=1}^{s}\frac{\partial^2\vec{r}_{ji}}{\partial\phi_q^2}\dot{\phi}_q^2, \tag{20}$$

$$\vec{a}_{ji,q}^{\epsilon} = \frac{\partial\vec{r}_{ji}}{\partial\phi_q}\ddot{\phi}_q. \tag{21}$$

From (15) and (21) the following relation yields

$$\frac{\vec{a}_{ji,q}^{\epsilon}}{\ddot{\phi}_q} = \frac{\vec{v}_{ji,q}}{\dot{\phi}_q}. \tag{22}$$

From (7) and (13)

$$\frac{\partial\phi_j}{\partial\phi_q} = \frac{\vec{\omega}_{j,q}}{\vec{\omega}_q} = \frac{\vec{\epsilon}_{j,q}^{\epsilon}}{\vec{\epsilon}_q}. \tag{23}$$

From (3)

$$\frac{\vec{\omega}_{j,q}}{\mu_{j,q}} = \frac{\vec{\omega}_q}{\mu_q}, \tag{24}$$

such that from (23) and (24)

$$\vec{\epsilon}_{j,q}^{\epsilon} = \frac{\vec{\omega}_{j,q}}{\vec{\omega}_q}\vec{\epsilon}_q = \frac{\mu_{j,q}}{\mu_q}\vec{\epsilon}_q, \tag{25}$$

which results in

$$\frac{\vec{\epsilon}_{j,q}^{\epsilon}}{\mu_{j,q}} = \frac{\vec{\epsilon}_q^{\epsilon}}{\mu_q}. \tag{26}$$

Based on (22) and (4) the following is obtained

$$\vec{a}_{ji,q}^{\epsilon} = \vec{v}_{ji,q}\frac{\ddot{\phi}_q}{\dot{\phi}_q} = \left[\frac{\vec{\omega}_q}{\mu_q},\vec{p}_{ji,q}^{*}\right]\frac{\ddot{\phi}_q}{\dot{\phi}_q} = \left[\frac{\vec{\omega}_q}{\dot{\phi}_q},\vec{p}_{ji,q}^{*}\right]\frac{\ddot{\phi}_q}{\mu_q} = \left[\vec{k},\vec{p}_{ji,q}^{*}\right]\frac{\ddot{\phi}_q}{\mu_q} = \left[\frac{\ddot{\phi}_q}{\mu_q}\vec{k},\vec{p}_{ji,q}^{*}\right] = \left[\frac{\vec{\epsilon}_q^{\epsilon}}{\mu_q},\vec{p}_{ji,q}^{*}\right], \tag{27}$$

where $\vec{k}$ is the unit vector perpendicular to the plane of the mechanism.

4. Moment of the force with respect to the pole of reduced mechanism

Let us assume that the **q**-th DOF of the planar mechanism with multiple DOF is enabled and all other DOF are disabled (the other variables remain constant). Additionally, let us assume that a force $\vec{\boldsymbol{F}}_{ji}$ is applied at the point $\boldsymbol{K}_{ji}$ of the mechanism (Fig. 8 shows only the **j**-th member). Then, the moment of the force $\vec{\boldsymbol{F}}_{\boldsymbol{ji}}$ for the instantaneous pole $\boldsymbol{P}_{\boldsymbol{j,q}}$ of the **j**-th member is

$$\vec{\boldsymbol{M}}_{\boldsymbol{ji,q}} = [\vec{\boldsymbol{p}}_{\boldsymbol{ji,q}}, \vec{\boldsymbol{F}}_{\boldsymbol{ji}}]. \tag{28}$$

If the same force is applied at the corresponding point K_{ji}^* of the partial reduced mechanism, which corresponds to the given degree of motion freedom **q** (Fig. 8b), then the moment of the force $\overset{P}{F}_{ji}$ with respect to the pole of the reduced mechanism $P_{ji,q}^*$

$$\vec{\boldsymbol{M}}_{\boldsymbol{ji,q}}^* = [\vec{\boldsymbol{p}}_{\boldsymbol{ji,q}}^*, \vec{\boldsymbol{F}}_{\boldsymbol{ji}}]. \tag{29}$$

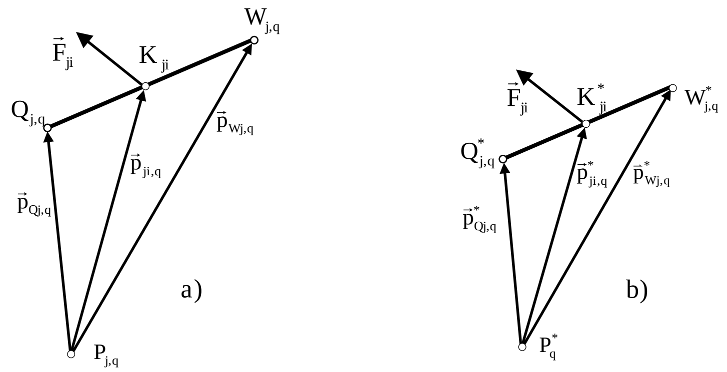

Fig. 8: The definition of the moment of the force with respect to the pole of the reduced mechanism

Vectors $\vec{M}^*{}_{ji,q}$ and $\vec{M}_{ji,q}$ are collinear and their relationship is

$$M_{ji,q}^* = [\vec{p}_{ji,q}^*, \vec{F}_{ji}] = [\mu_{j,q}\vec{p}_{ji,q}, \vec{F}_{ji}] = \mu_{j,q}[\vec{p}_{ji,q}, \vec{F}_{ji}], \tag{30}$$

i.e.

$$\vec{\boldsymbol{M}}_{\boldsymbol{ji,q}}^* = \boldsymbol{\mu}_{\boldsymbol{j,q}}\vec{\boldsymbol{M}}_{\boldsymbol{ji,q}}. \tag{31}$$

5. Equation of the mechanism motion expressed via the moment of the reduced mechanism

Let us assume that a planar mechanism has s degrees of motion freedom. Additionally, let us assume that the mechanism has the q-th degree enabled and all other degrees are disabled. The moment of the partial reduced mechanism corresponding to the virtual displacement q, as a consequence of applied external forces, is defined as the sum of moments of all external forces associated with and applied to the reduced mechanism, with respect to its pole P_q^*

$$\vec{M}_{R,q}^{*,ext} = \sum_j \sum_i \vec{M}_{ji,q}^{*,ext}, \tag{32}$$

i.e.

$$\vec{\boldsymbol{M}}_{\boldsymbol{R,q}}^{*,ext} = \sum_{\boldsymbol{j}} \sum_{\boldsymbol{i}} \boldsymbol{\mu}_{\boldsymbol{j,q}} \vec{\boldsymbol{M}}_{\boldsymbol{ji,q}}^{\boldsymbol{ext}} = \sum_{\boldsymbol{j}} \boldsymbol{\mu}_{\boldsymbol{j,q}} \sum_{\boldsymbol{i}} \vec{\boldsymbol{M}}_{\boldsymbol{ji,q}}^{\boldsymbol{ext}}. \tag{33}$$

Similar case is when inertial forces are concerned. Let us define the moment of the partial reduced mechanism for the virtual displacement **q**, yielding from the inertial loading, as the sum of moments of inertial forces associated with and applied to the reduced mechanism, with respect to its pole P_q^*

$$\vec{M}_{R,q}^{*,in} = \sum_j \sum_i \vec{M}_{ji}^{*,in} = \sum_j \mu_{j,q} \sum_i \vec{M}_{ji}^{in}. \tag{34}$$

Variation of the mechanical work done by applied external forces on the displacement **q** is

$$\boldsymbol{\delta A^{ext}} = \sum_{\boldsymbol{q}} \sum_{\boldsymbol{j}} \sum_{\boldsymbol{i}} (\vec{\boldsymbol{F}}_{\boldsymbol{ji}}^{\boldsymbol{ext}}, \boldsymbol{\delta}\vec{\boldsymbol{s}}_{\boldsymbol{ji,q}}) = \sum_{\boldsymbol{q}} \sum_{\boldsymbol{j}} \sum_{\boldsymbol{i}} (\vec{\boldsymbol{F}}_{\boldsymbol{ji}}^{\boldsymbol{ext}}, \vec{\boldsymbol{v}}_{\boldsymbol{ji,q}} \boldsymbol{\delta t}) = \boldsymbol{\delta t} \sum_{\boldsymbol{q}} \sum_{\boldsymbol{j}} \sum_{\boldsymbol{i}} (\vec{\boldsymbol{F}}_{\boldsymbol{ji}}^{\boldsymbol{ext}}, \vec{\boldsymbol{v}}_{\boldsymbol{ji,q}}), \tag{35}$$

where,

$\delta\vec{s}_{ji,q}$ - variation of the point K_{ji} displacement resulting from the displacement **q**,
$\vec{v}_{ji,q}$ - velocity of the point K_{ji} resulting from the **q**-th velocity, and
δt - time variation.

Considering that for a set of three arbitrary vectors the following holds

$$(\vec{a}, [\vec{b}, \vec{c}]) = (\vec{b}, [\vec{c}, \vec{a}]), \tag{36}$$

then we can write

$$(\vec{F}_{ji}^{ext}, \vec{v}_{ji,q}) = (\vec{F}_{ji}^{ext}, \vec{v}_{ji,q}) = (\vec{F}_{ji}^{ext}, [\vec{\omega}_{j,q}, \vec{p}_{ji,q}]) = (\vec{\omega}_{j,q}, [\vec{p}_{ji,q}, \vec{F}_{ji}^{ext}]) = (\vec{\omega}_{j,q}, \vec{M}_{ji,q}^{ext}). \tag{37}$$

From (3), we obtain

$$\vec{\omega}_{j,q} = \mu_{j,q} \frac{\vec{\omega}_q}{\mu_q}. \tag{38}$$

Based on (35), it implies

$$\delta A^{ext} = \delta t \sum_q \sum_j \sum_i (\vec{M}_{ji,q}^{ext}, \vec{\omega}_{j,q}) = \delta t \sum_q \sum_j \sum_i \left(\vec{M}_{ji,q}^{ext}, \mu_{j,q} \frac{\vec{\omega}_q}{\mu_q}\right) = \delta t \sum_q \sum_j \sum_i \left(\mu_{j,q} \vec{M}_{ji,q}^{ext}, \frac{\vec{\omega}_q}{\mu_q}\right) = \delta t \sum_q \sum_j \sum_i \left(\vec{M}_{ji,q}^{*,ext}, \frac{\vec{\omega}_q}{\mu_q}\right) = \delta t \sum_q \left(\sum_j \sum_i \vec{M}_{ji,q}^{*,ext}, \frac{\vec{\omega}_q}{\mu_q}\right). \tag{39}$$

From (39) and (32) we obtain

$$\delta A^{ext} = \delta t \sum_q \left(\vec{M}^{*,ext}_{R,q}, \frac{\vec{\omega}_q}{\mu_q} \right). \tag{40}$$

If we start from the general equation of dynamics (Lagrange-D'alembert principle), then

$$\boldsymbol{\delta A^{ext} + \delta A^{in,\omega} + \delta A^{in,\epsilon} = 0}, \tag{41}$$

such that, analog to (40)

$$\boldsymbol{\delta A^{in,\omega} = \delta t \sum_q \left(\vec{M}^{*,in,\omega}_{R,q}, \frac{\vec{\omega}_q}{\mu_q} \right)}, \qquad \boldsymbol{\delta A^{in,\epsilon} = \delta t \sum_q \left(\vec{M}^{*,in,\epsilon}_{R,q}, \frac{\vec{\omega}_q}{\mu_q} \right)}. \tag{42}$$

Now, based on (40-42)

$$\delta t \sum_q \left(\vec{M}^{*,ext}_{R,q}, \frac{\vec{\omega}_q}{\mu_q} \right) + \delta t \sum_q \left(\vec{M}^{*,in,\omega}_{R,q}, \frac{\vec{\omega}_q}{\mu_q} \right) + \delta t \sum_q \left(\vec{M}^{*,in,\epsilon}_{R,q}, \frac{\vec{\omega}_q}{\mu_q} \right) = 0, \tag{43}$$

i.e.

$$\delta t \sum_q \left(\left(\vec{M}^{*,ext}_{R,q} + \vec{M}^{*,in,\omega}_{R,q} + \vec{M}^{*,in,\epsilon}_{R,q} \right), \frac{\vec{\omega}_q}{\mu_q} \right) = 0. \tag{44}$$

Since the variation of time $\delta t \neq 0$, then

$$\sum_q \left(\left(\vec{M}^{*,ext}_{R,q} + \vec{M}^{*,in,\omega}_{R,q} + \vec{M}^{*,in,\epsilon}_{R,q} \right), \frac{\vec{\omega}_q}{\mu_q} \right) = 0. \tag{45}$$

Expression (45) will be satisfied for any kinematic condition of the mechanism if and only if

$$\left(\vec{M}^{*,ext}_{R,q} + \vec{M}^{*,in,\omega}_{R,q} + \vec{M}^{*,in,\epsilon}_{R,q} \right), \frac{\vec{\omega}_q}{\mu_q} = 0, \ (q = 1,2,3, \dots, s). \tag{46}$$

6. Moment of inertia of the partial reduced mechanism

Let us assume that any point K^*_{ji} of the **j**-th member of the **q**-th partial reduced mechanism has the mass m_{ji} , which corresponds to the point K_{ji} of the real mechanism with multiple DOF (Fig. 9). Consequently, the mass of the **j**-th member of the reduced mechanism will be equal to the mass of the **j**-th member of the real mechanism, but the corresponding moments of inertia, in general, will not be equal.

The moment of inertia of **j**-th member of the reduced mechanism, whose real mechanism has multiple DOF, for its pole $P^*_{j,q}$, is

$$J^*_{j,q} = \sum_i m_{ji} \left(p^*_{ji,q} \right)^2 = \sum_1 m_{ji} \left(\mu_{j,q} p_{ji,q} \right)^2 = \mu^2_{j,q} \sum_i m_{ji} \left(p_{ji,q} \right)^2 = \mu^2_{j,q} J_{j,q}, \tag{47}$$

where,

$p^*_{ji,q} = \overrightarrow{P^*_{j,q} K^*_{ji}}$ - the relative radius vector between the pole P^*_j and the point K^*_{ji},

$J_{j,q}$ - moment of inertia of **j**-th member of the real mechanism for the pole $P_{j,q}$.

The moment of inertia of the partial reduced mechanism is defined as the moment of inertia of that mechanism with respect to its pole P^*_q, such that

$$J^*_q = \sum_j J^*_{j,q} = \sum_j \mu^2_{j,q} J_{j,q}. \tag{48}$$

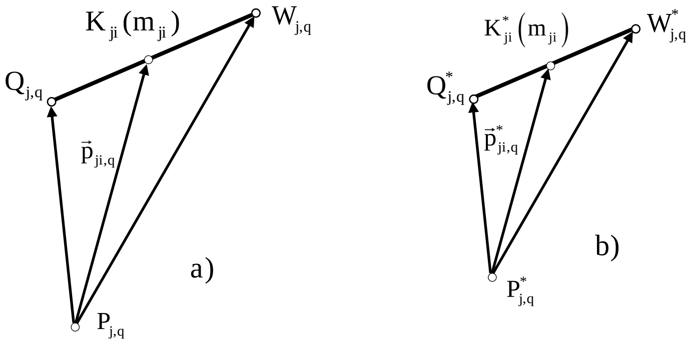

Fig. 9: The definition of the reduced mechanism's moment of inertia

7. Angular acceleration determination of driving members for a mechanism with multiple DOF

Let us assume that a planar mechanism with multiple DOF performs a motion caused by a set of applied external forces. The third member on the left hand-side in (46) is the moment of the **q**-th reduced mechanism resulting from secondary inertial forces

$$\vec{M}^{*,in,\epsilon}_{R,q} = \sum_j \sum_i \vec{M}^{*,in,\epsilon}_{ji,q} = \sum_j \sum_i \left[\vec{p}^*_{ji,q}, \vec{F}^{in,\epsilon}_{ji} \right] = - \sum_j \sum_i m_{ji} \left[\vec{p}^*_{ji,q}, \vec{a}^{in,\epsilon}_{ji} \right] = - \sum_j \sum_i m_{ji} \left[\vec{p}^*_{ji,q}, \left[\frac{\vec{\epsilon}^{\epsilon}_q}{\mu_q}, \vec{p}^*_{ji,q} \right] \right]. \tag{49}$$

Since the following hold for the double cross vector product

$$\left[\vec{a}, \left[\vec{b}, \vec{c} \right] \right] = \vec{b}(\vec{c}, \vec{a}) - \vec{c}(\vec{a}, \vec{b}), \tag{50}$$

then

$$\vec{M}^{*,in,\epsilon}_{R,q} = - \sum_j \sum_i m_{ji} \left[\frac{\vec{\epsilon}^{\epsilon}_q}{\mu_q} \left(\vec{p}^*_{ji}, \vec{p}^*_{ji} \right) - \vec{p}^*_{ji} \left(\vec{p}^*_{ji}, \frac{\vec{\epsilon}^*_q}{\mu_q} \right) \right]. \tag{51}$$

Since $\vec{p}^*_{ji} \perp \vec{\epsilon}^{\epsilon}_1$, then $\left(\vec{p}^*_{ji}, \frac{\vec{\epsilon}^{\epsilon}_q}{\mu_q} \right) = 0$, such that

$$\vec{M}^{*,in,\epsilon}_{R,q} = - \sum_j \sum_i m_{ji} \frac{\vec{\epsilon}^{\epsilon}_q}{\mu_q} \left(\vec{p}^*_{ji}, \vec{p}^*_{ji} \right) = - \sum_j \sum_i m_{ji} \frac{\vec{\epsilon}^{\epsilon}_q}{\mu_q} \left(p^*_{ji} \right)^2 = - \frac{\vec{\epsilon}^{\epsilon}_q}{\mu_q} \sum_j \sum_i m_{ji} \left(p^*_{ji} \right)^2 = - \frac{\vec{\epsilon}^{\epsilon}_q}{\mu_q} J^*_q. \tag{52}$$

Then, from (46) and (52) the following yields

$$\vec{\epsilon}^{\epsilon}_q = \vec{\epsilon}_q = \frac{\vec{M}^{*,ext}_{R,q} + \vec{M}^{*,in,\omega}_{R,q}}{J^*_q}\mu_q, \qquad (q = 1,2,3,\dots,s). \qquad (53)$$

From (53), the angular accelerations can be calculated for driving members of the mechanism with multiple DOF, using the partial reduction of the mechanism.

8. Example

Fig. 10 shows the mechanism with two DOF. Moment $M_1 = 200\ Nm$ is applied on the member 1 and the moment $M_4 = 300\ Nm$ is applied on the member 4. At the start, angular velocities for members 1 and 4 are $\omega_1 = 40\ {rad}/_{s}$ and $\omega_4 = 40\ {rad}/_{s}$, respectively. The following geometrical data are given:

$$\overline{AB} = 3L, \overline{BD} = \overline{DE} = \overline{EH} = 3L, \overline{DC_3} = 2L, L = 0.1\ m.$$

Moments of inertia of the mechanism members are:

$$J_{1A} = 0.08\ kgm^2, J_{C3} = 0.05\ kgm^2, J_{H4} = 0.2\ kgm^2, mass\ m_3 = 4\ kg.$$

The mass of the member 2 can be neglected. Applying the method of partial reduction, determine angular accelerations of driving members.

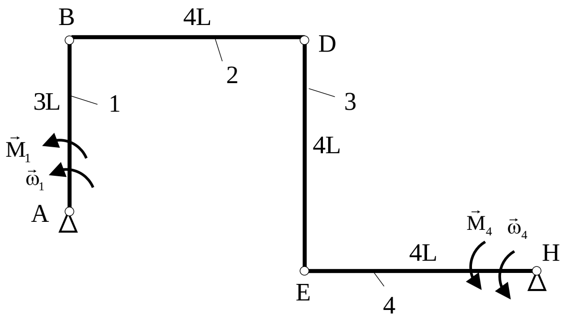

Fig. 10: The mechanism in the example

8.1. Solution using the classical method

The velocity of the point E, expressed using the velocity of the point B, is (Fig. 11)

$$\vec{v}_E = \vec{v}_B + \vec{v}_{D,B} + \vec{v}_{E,D}. \qquad (54)$$

By projecting (54) onto x axis, the following is obtained

$$0 = -v_B + v_{ED},$$
$$0 = -3L\omega_1 + 4L\omega_3,$$
$$\omega_3 = \frac{3}{4}\omega_1 = \frac{3}{4}\cdot 40\,\frac{\text{rad}}{\text{s}} = 30\,\frac{\text{rad}}{\text{s}}. \qquad (55)$$

Projection of (54) onto y axis is

$$-v_E = v_{DB},$$
$$-4L\omega_4 = 4L\omega_2,$$
$$\omega_2 = -\omega_2 = -\omega_4 = -60\,\frac{\text{rad}}{\text{s}}. \qquad (56)$$

Acceleration of the point E, expressed using the acceleration of the point B, is

$$\vec{a}^n_E + \vec{a}^t_E = \vec{a}^n_B + \vec{a}^t_B + \vec{a}^{rot}_{D,B} + \vec{a}^{axp}_{D,B} + \vec{a}^{rot}_{E,D} + \vec{a}^{axp}_{E,D}. \qquad (57)$$

By projecting (57) onto x axis, the following is obtained

$$a^n_E = -a^t_B - a^{axp}_{D,B} + a^{rot}_{E,D}\,, \qquad (58)$$
$$4L\omega_4^2 = -3L\epsilon_1 - 4L\omega_2^2 + 4L\epsilon_3,$$
$$4\omega_4^2 = -3\epsilon_1 - 4\omega_2^2 + 4\epsilon_3,$$
$$4\cdot 1600\,\frac{\text{rad}}{\text{s}^2} = -3\epsilon_1 - 4\cdot 3600\,\frac{\text{rad}}{\text{s}^2} + 4\epsilon_3,$$
$$-3\epsilon_1 + 4\epsilon_3 = 20800\,\frac{\text{rad}}{\text{s}^2}. \qquad (59)$$

By projecting (57) onto y axis, the following is obtained

$$-a^t_E = -a^n_B + a^{rot}_{D,B} + a^{axp}_{E,D}\,, \qquad (60)$$
$$-4L\epsilon_4 = -3L\omega_1^2 + 4L\epsilon_2 + 4L\omega_3^2,$$
$$-4\epsilon_4 = -3\omega_1^2 + 4\epsilon_2 + 4\omega_3^2,$$
$$-4\epsilon_4 = -3\cdot 1600\,\frac{\text{rad}}{\text{s}^2} + 4\epsilon_2 + 4\cdot 900\,\frac{\text{rad}}{\text{s}^2},$$
$$\epsilon_4 + \epsilon_2 = 300\,\frac{\text{rad}}{\text{s}^2}. \qquad (61)$$

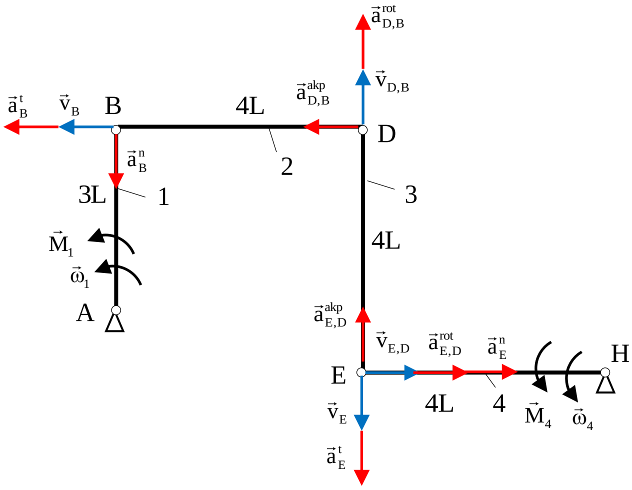

Fig. 11: Determination of kinematic characteristics of the mechanism in the example

By applying the law of change of angular momentum (Hibbeler, 2013) for the point A of the member 1 (Fig. 12), the following is obtained

$$\Sigma M_A = 0\,,$$
$$M_1 - F_B\cdot 3L = J_{1A}\epsilon_1,$$

$$200\ \mathrm{Nm} - \mathrm{F_B}(0{,}3\ \mathrm{m}) = (0{,}08\ \mathrm{kgm^2})\,\epsilon_1. \qquad (62)$$

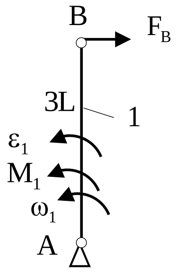

Fig. 12: Application of the angular momentum change law for the point A of the member 1

Since the member 2 has negligible mass, then the force F_B is transferred from the joint B to the joint D, as well (Fig. 13).

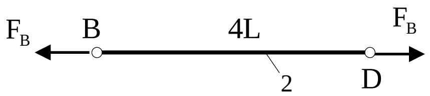

Fig. 13: The force $\mathrm{F_B}$ transferred along the rod 2

By applying the law of angular momentum change for the center of mass of the member 3, the following is obtained (Fig. 14)

$$\mathrm{F_B}2\mathrm{L} + \mathrm{X_E}2\mathrm{L} = \mathrm{J_{3c}}\epsilon_3,$$
$$\mathrm{F_B}(0{,}2\mathrm{m}) + \mathrm{X_E}(0{,}2\mathrm{m}) = (0{,}05\mathrm{kgm^2})\epsilon_3,$$
$$\mathrm{F_B}(4\mathrm{m}) + \mathrm{X_E}(4\mathrm{m}) = (1\mathrm{kgm^2})\epsilon_3. \qquad (63)$$

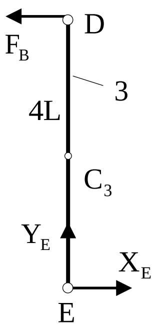

Fig. 14: Law of motion application for the beam 3

By applying the law of center of mass motion in the *x* direction for the member 3, the following is obtained

$$\mathrm{m_3}\ddot{\mathrm{x}}_{\mathrm{C3}} = \mathrm{X_E} - \mathrm{F_B},$$
$$(4\mathrm{kg})\ddot{\mathrm{x}}_{\mathrm{C3}} = \mathrm{X_E} - \mathrm{F_B}.$$

Since $\ddot{\mathrm{x}}_{\mathrm{C3}} = 4\mathrm{L}\omega_4^2 - \epsilon_3 2\mathrm{L}$, then

$$4(4\mathrm{L}\omega_4^2 - \epsilon_3 2\mathrm{L}) = \mathrm{X_E} - \mathrm{F_B},$$
$$(16\mathrm{kg})\ \mathrm{L}\omega_4^2 - (8\mathrm{kg})\ \epsilon_3\mathrm{L} = \mathrm{X_E} - \mathrm{F_B},$$
$$(5760\mathrm{N}) - (0{,}8\mathrm{kgm})\epsilon_3 = \mathrm{X_E} - \mathrm{F_B}. \qquad (64)$$

By applying the law of center of mass motion in the *y* direction for the member 3, the following is obtained

$$\mathrm{m_3}\ddot{\mathrm{y}}_{\mathrm{C3}} = \mathrm{Y_E},$$
$$\ddot{\mathrm{y}}_{\mathrm{C3}} = -4\mathrm{L}\epsilon_4 - 2\mathrm{L}\omega_3^2,$$
$$\mathrm{m_3}(-4\mathrm{L}\epsilon_4 - 2\mathrm{L}\omega_3^2) = \mathrm{Y_E},$$
$$(4\mathrm{kg})(-0{,}4\epsilon_4 - 0{,}2\cdot 900) = \mathrm{Y_E},$$
$$-(1{,}6\mathrm{kgm})\epsilon_4 - 720\ \mathrm{N} = \mathrm{Y_E}. \qquad (65)$$

By applying the angular momentum change law for the pole H of the member 4 (Fig. 15), the following is obtained

$$\mathrm{Y_E}4\mathrm{L} + \mathrm{M_4} = \mathrm{J_{4H}}\epsilon_4,$$
$$\mathrm{Y_E}(0{,}4\ \mathrm{m}) + 300\ \mathrm{Nm} = (0{,}2\ \mathrm{kgm^2})\epsilon_4. \qquad (66)$$

Unknowns are: $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, F_B, X_E$, and Y_E.

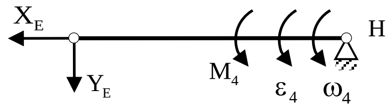

Fig. 15: Law of angular momentum change application for the point H of the member 4

From equation (59) and (61-66), the following is obtained

$\epsilon_1 = 1236{,}5930\,\frac{\mathrm{rad}}{\mathrm{s^2}}$, $\epsilon_2 = 285{,}7143\,\frac{\mathrm{rad}}{\mathrm{s^2}}$, $\epsilon_3 = 6127{,}4447\,\frac{\mathrm{rad}}{\mathrm{s^2}}$, $\epsilon_4 = 14{,}2857\,\frac{\mathrm{rad}}{\mathrm{s^2}}$, $Y_E = -742{,}8571\ \mathrm{N}$, $F_B = 336{,}9085\ \mathrm{N}$, $X_E = 1194{,}9526\ \mathrm{N}$.

8.2. Solution using partial reduction of the mechanism

a) The first partial reduced mechanism that will be analyzed is the case with the real mechanism driving member 4 blocked $(\delta\phi_1 \neq 0, \delta\phi_4 = 0)$. The reduced mechanism for that case is shown in Fig. 16. The pole of the reduced mechanism is denoted by P_I^*.

Since in this case, based on (59): $-3\epsilon_1 + 4\epsilon_3 = 20800\,\frac{\mathrm{rad}}{\mathrm{s^2}}$, then the primary angular acceleration of the member 3, which is independent of the angular acceleration of member 1, is

$$4\epsilon_3^{\omega} = 20800\,\frac{\mathrm{rad}}{\mathrm{s^2}}\ \text{, i.e. } \epsilon_3^{\omega} = 5200\,\frac{rad}{s^2}. \qquad (67)$$

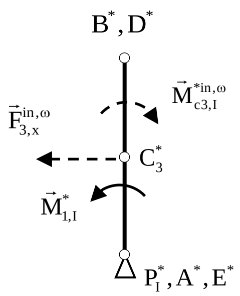

Fig. 16: Partial reduced mechanism when the member 4 is blocked

If the factor of the reduction of member 1 is taken $\mu_1 = 1$, then

$$\mu_3 = \frac{\overrightarrow{P_I^* B^*}}{\overrightarrow{ED}} = \frac{3}{4} = 0{,}75. \qquad (68)$$

Further

$$\ddot{x}_{C3} = 4L\omega_4^2 - \epsilon_3 2L,$$

$$\ddot{x}^{*}_{C3} = 4L\omega_4^2 - \epsilon_3^{\omega}2L = (4 \cdot 0{,}1\,3600 - 5200 \cdot 2 \cdot 0{,}1)\frac{\mathrm{m}}{\mathrm{s}^2} = 400\frac{\mathrm{m}}{\mathrm{s}^2}. \quad (69)$$

The principal vector of the primary inertial forces of the member 3, with the center of mass in the point C_3, is

$$F_{3x}^{in,\omega} = -m_3\ddot{x}_{C3} = -1600\ \mathrm{N}. \quad (70)$$

Moment of primary inertial forces of the member 3 for its center of mass is

$$\mathrm{M}_{C3}^{in,\omega} = \mathrm{J}_{C3}\,\epsilon_3^{\omega} = (0{,}05 \cdot 5200)\mathrm{Nm} = 260\ \mathrm{Nm}. \quad (71)$$

The moment of inertia of the member 3 for the point E is

$$\mathrm{J}_{E3} = \mathrm{J}_{C3} + \mathrm{m}_3(2\mathrm{L})^2 = (0{,}05 + 4 \cdot 0{,}04)\mathrm{kgm}^2 = 0{,}21\ \mathrm{kgm}^2, \quad (72)$$

and the moment of inertia of the first partial reduced mechanism, according to (48), is

$$J_{\mathrm{I}}^{*} = \mu_1^2 J_{A1} + \mu_3^2 J_{E3},$$
$$J_{\mathrm{I}}^{*} = (1 \cdot 0{,}08 + 0{,}75^2 \cdot 0{,}21)\mathrm{kgm}^2 = 0{,}1981\ \mathrm{kgm}^2. \quad (73)$$

In this way the angular acceleration of the member 1 is

$$\epsilon_1 = \frac{M_{1,\mathrm{I}}^{*} - M_{C3,\mathrm{I}}^{*,in,\omega} + F_3^{in,\omega}2L\mu_3}{J_{\mathrm{I}}^{*}}\mu_1 = \frac{\mu_1 M_1 - \mu_3 M_{C3}^{in,\omega} + F_3^{in,\omega}2L\mu_3}{J_{\mathrm{I}}^{*}}\mu_1, \quad (74)$$

i.e.

$$\epsilon_1 = \frac{200 - 0{,}75 \cdot 260 + 1600 \cdot 0{,}2 \cdot 0{,}75}{0{,}1981}1 = 1236{,}7491\frac{rad}{s^2}. \quad (75)$$

b) Now, we will consider the case of the partial reduced mechanism when the driving member 1 of the real mechanism is blocked $(\delta\phi_1 = 0, \delta\phi_4 \neq 0)$.

Reduced mechanism for that case is shown in Fig. 17. Pole of the reduced mechanism is denoted by $\mathrm{P}_{\mathrm{II}}^{*}$.

Since in this case

$$\omega_3 = \frac{3}{4}\omega_1 = 0, \quad (76)$$

then

$$\mu_{3,\mathrm{II}} = 0. \quad (77)$$

Further

$$\ddot{y}_{C3} = -4\mathrm{L}\epsilon_4 - 2\mathrm{L}\omega_3^2,$$
$$\ddot{y}_{C3}^{\omega} = -2\mathrm{L}\omega_3^2 = -(2 \cdot 0{,}1 \cdot 900)\frac{\mathrm{m}}{\mathrm{s}^2} = -180\frac{\mathrm{m}}{\mathrm{s}^2}, \quad (78)$$
$$\mathrm{F}_{C3}^{in,\omega} = -\mathrm{m}_3\ddot{y}_{C3}^{\omega} = 4 \cdot (180)\mathrm{N} = 720\ \mathrm{N}. \quad (79)$$

If the reduction factor $\mu_4 = 1$ is taken, then

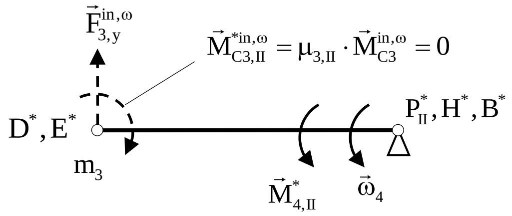

Fig. 17. Partial reduced mechanism when the member 1 is blocked

$$\overline{P_{\mathrm{II}}^{*}D^{*}} = \mu_4 4L = 4L = 0{,}4\ \mathrm{m},$$
$$J_{II}^{*} = \mu_4^2 J_{4H} + m_3\overline{P_2^{*}D^{*}}^2,$$
$$J_{II}^{*} = (1^2 \cdot 0{,}2 + 4 \cdot 0{,}4^2)\mathrm{kgm}^2 = 0{,}84\ \mathrm{kgm}^2. \quad (80)$$

Angular acceleration of the member 4 is

$$\epsilon_4 = \frac{M_{4,II}^{*} - F_{C3,y}^{in,\omega}\cdot\overline{P_2^{*}D^{*}}}{J_{II}^{*}}\mu_4 = \frac{\mu_4 M_4 - F_{C3,y}^{in,\omega}\cdot\overline{P_2^{*}D^{*}}}{J_{II}^{*}}\mu_4, \quad (81)$$
$$\epsilon_4 = \left(\frac{1 \cdot 300 - 720 \cdot 0{,}4}{0{,}84} \cdot 1\right)\frac{\mathrm{rad}}{\mathrm{s}^2} = 14{,}2857\frac{\mathrm{rad}}{\mathrm{s}^2}. \quad (82)$$

Therefore, the solutions using the proposed method and the classical method are same.

9. Conclusion

In contrary to the application of the general laws of multibody system dynamics, where as the result of solution to an inverse problem of dynamics, a set of coupled equations is obtained, by applying the partial reduction of mechanism, solutions are obtained in a simpler way. Specifically, in this example there were seven coupled equations obtained using the classical method, but only two decoupled equations with one unknown per equation using the proposed method of partial reduced mechanism. Comparing the obtained results using different methods the differences are negligible and they exist only due to the round off errors.

References

Gernet MM (1973). Kurs teoretičeskoj mehaniki. Višaja škola, Moscow, Russia.

Hibbeler RC (2013). Engineering mechanics: Statics and dynamics. Prentice-Hall, Inc., Upper Saddle River, New Jersey, USA.

Hufnagl B (1984). Mehanizmi, Univerzitetski udžbenik. Mašinski fakultet, Sarajevo, Bosnia and Herzegovina.

Ilic B (1968). Mehanizmi III, Komparativna obrada mehanizama. Naučna knjiga, Belgrade, Serbia.

Ilic B (1965). Mehanizmi, Reducirani mehanizam. Zavod za izdavanje udžbenika Socijalističke, Belgrade, Serbia.

Johnston ER, Beer F, and Eisenberg E (2009). Vector mechanics for engineers: Statics and dynamics. McGraw-Hill, New Jersey, USA.